RECURSIVE DYNAMIC PROGRAMMING: HEURISTIC RULES, BOUNDING AND STATE SPACE REDUCTION

Henrik Kure

*Dina, Danish Informatics Network In the Agricultural Sciences
Royal Veterinary and Agricultural University
Bulowsvej 13, DK-1870 Frb., Copenhagen. E-mail: kure@dina.kvl.dk*

Abstract: An alternative approach towards dynamic programming (DP) is presented: Recursions. A basic deterministic model and solution function are defined and the model is generalized to include stochastic processes, with the traditional stochastic DP model as a special case. Heuristic rules are included in the models simply as restrictions on decision spaces and a small slaughter pig marketing planning example illustrates the potential state space reductions induced by such rules.

Keywords: Dynamic programming, recursive algorithms, heuristics, stochastic modelling.

1. INTRODUCTION

Dynamic Programming (DP) is a general optimization principle, formulated by Bellman (1957). The principle is usually related to the field of Operations Research (OR), but the underlying and rather intuitive "Principle of optimality" (Bellman, 1957) is essential to several specific optimization techniques, including AI algorithms as A^* (Winston, 1992) and the solution methods of *Influence Diagrams* and *Valuation Networks* (Shenoy, 1992).

Value Iteration (VI) is (at least in OR) the best known and widely used DP technique. In this paper an alternative and more general approach towards DP is presented: *Recursions*. The approach (Recursive Dynamic Programming, RDP) is basically a direct implementation of the fundamental functional equation of DP (as defined by eg. Nemhauser (1966)):

$$f_n(x_n) = \min_{d_n \in D_n} [r_n(x_n, d_n) + f_{n-1}(t_n(x_n, d_n))] \quad (1)$$

where n is the stage number,
 x_n is the state ($n = 0$ indicates a goal state),
 f_n is the solution function at stage n . Returns the solution to the problem associated with x_n ($f_0(x_0)$ is per definition a known value),

d_n is a decision, chosen from the set of valid decisions, D_n ,

r_n is the gain function and

t_n is the transition function.

It must be emphasized that this approach neither is nor intend to be novel, but it appears to be relatively unknown to OR and agricultural scientists (Kure, 1995). The purpose of this paper is to show the simplicity and the potential computational benefits of RDP.

The approach has, as will be shown, several theoretical and applicational implications. The methods presented may be applied to virtual any problem that might be solved using VI, including agricultural planning/optimization problems like facility management, replacement problems and farm operations scheduling. The methods (as defined in this paper) have been applied to simple illustrative problems and a slaughter pig marketing decision support system which is under construction.

All models and functions in this paper are specified and defined using the syntax of the specification language, VDM (see eg. Jones (1990) for an introduction) which in many aspects is equivalent to functional languages as SML.

2. RECURSIVE DYNAMIC PROGRAMMING

An example might illustrate the model, the problem, the method and some implications of RDP:

Consider a road map represented as a *directed graph* as shown in fig. 1. Each *node* (or *state*) in the graph represents a crossroads and each (directed) *edge* represents a piece of one-way road. A *decision* prescribes what edge to follow from a given node. Node A is defined as the *start node*. Node B has no *successors*; it is a *goal node*. Edges might be interpreted as a pair of functions:

- a) A *transformation function*:
 $t: State \times Decision \rightarrow State$
 returning a new node/state and
- b) a *gain function*:
 $g: State \times Decision \rightarrow Gain$
 returning a *gain* (ie. the length of an edge).

A *policy* is (in this presentation) defined as an assignment of a (valid) decision to every node in a sub-set of nodes in the graph and a *complete policy* assigns a decision to every node (except goal nodes) in the graph. The *optimization problem* to be considered here is: *What is the shortest path from A to B?*

2.1 The Data Model

Based on these basic components a deterministic data model/data type of a system is formulated as a finite mapping from states into a finite mapping from decisions into gains and (succeeding) states:

$$DPMModel = State \mapsto Decision \mapsto Gain \times State \\ inv(dpm) \dots$$

where the invariant must restrict the model to be finite and cycle free (see Kure (1995) for a definition). The model will later be modified to be based on functions instead of finite mappings. In addition to the data model of the system a representation of a policy is required:

$$Policy = State \mapsto Decision$$

Fig. 1. A road map.

2.2 The Problem

In order to formulate and solve the problem, a set of basic functions is *specified* (see Jones (1990)):

complete_pol (*dpm*: *DPMModel*, *p*: *Policy*) *r*: boolean
 post $r = (\text{dom } p = \text{dom } dpm \wedge \forall st \in \text{dom } p \cdot p(st) \in \text{dom } dpm(st))$

all_pol (*dpm*: *DPMModel*) *r*: *Policy*-set
 post $r = \{p \in Policy \mid complete_pol(dpm, p)\}$

combine: *Gain* $\times$ *Gain* $\rightarrow$ *Gain*

all_pol returns all complete policies applicable to a given instance, *dpm*, of the model. The function *comb* is not yet defined (in the example it is defined as a simple addition of the arguments), but is used in the *definition* (see Jones (1990) for an introduction to direct definitions in VDM) of the objective function:

objfct: *DPMModel* $\times$ *Policy* $\times$ *State* $\rightarrow$ *Gain*
objfct (*dpm*, *p*, *st*) $\triangleq$
 let (*g*, *n_st*) = *dpm* (*st*)(*p* (*st*))
 in if $n_st \notin \text{dom } p$ then *g*
 else *comb* (*g*, *objfct* (*dpm*, *p*, *n_st*))

Given an instance of the model (*dpm*: *DPMModel*) and a state (*st*: *State*), the *problem* is to minimize the objective function *objfct* with respect to all complete policies ($p \in all_pol(dpm)$):

$$\min \{ objfct(dpm, p, st) \mid p \in all_pol(dpm) \}$$

In other words the goal is to find a *solution function*, *solution*, that satisfies the specification:

solution (*dpm*: *DPMModel*, *st*: *State*) *r*: *Gain*
 pre $st \in \text{dom } dpm$
 post $r = \min \{ objfct(dpm, p, st) \mid p \in all_pol(dpm) \}$

The *state space* is defined as the total set of states (except goal states) in the model and a *decision space* as the total set of valid decisions in a given state:

state_space (*dpm*) $\triangleq \text{dom } dpm$
decision_space (*dpm*, *st*) $\triangleq \text{dom } dpm(st)$

2.3 A solution function

By introducing two new functions a solution function may be derived (see Kure (1995) where a proof based on a mathematical interpretation of Bellman's Principle of Optimality is given):

min_two: *Y* $\times$ *Y* $\rightarrow$ *Y*
min_two (*a*, *b*) $\triangleq$ if $a < b$ then *a* else *b*

```

minimum:  $(X \rightarrow Y) \times X\text{-set} \rightarrow Y$ 
minimum (g, as)  $\triangle$ 
  let l_min (m, l_as) =
    if l_as = {} then m else
      let a  $\in$  l_as
        in l_min (min_two (m, g (a)), l_as - {a})
  and fst_a  $\in$  as
  in l_min (g (fst_a), as - {fst_a})
  pre as  $\neq$  {}

```

```

solution:  $DPMModel \times State \rightarrow Gain$ 
solution (dpm, st)  $\triangle$ 
  let f (d) =
    let (g, n_st) = dpm (st)(d)
    in if n_st  $\in$  dom dpm then g
      else comb (g, solution (dpm, n_st))
  in minimum (f, dom dpm (st))
  pre st  $\in$  dom dpm

```

The function *minimum* minimizes the returns of the function *g* over *as*, a finite set of arguments to *g*. In *solution minimum* is utilized to minimize *f* over the finite set of valid decisions, *dom dpm (st)*, where *f* returns the optimal gain of executing a decision, *d*. The problem of finding the shortest path from A to B in fig. 1 is solved recursively starting in the start state (A).

2.4 Memory functions

As shown in the example, RDP consists of a recursively decomposition of a problem into and solving of sub-problems, but the solution function, *solution*, does not prevent the same problem to be solved more than once (ie. in the example different paths may lead to the same node). This (possibly inefficient) resolving of the same problem may be eliminated by keeping a record of known solutions: If a solution is recorded, it is already known and may be used directly. Otherwise the problem must be solved and the result saved in the record for later use. This use of record keeping or use of a *memory function* (Brassard and Bratley, 1988) is an (or *the*) essential part of DP and may result in great savings in computing time.

The record of solutions might be represented as a map:

$$Solutions = State \mapsto Gain$$

If knowledge of the policy leading to the solution is required, this *optimal policy* might be represented as an extension to the record of solutions:

$$SolutionsOptPolicy = State \mapsto Gain \times Decision$$

All solution functions in this paper might be extended with memory functions (see Kure (1995) for direct definitions).

2.5 Function based model

The basic data model may be modified to be based on functions instead of (finite) maps:

```

NotInDomain = State  $\rightarrow$  boolean
DecisionSpace = State  $\rightarrow$  Decision-set
Consq = State  $\times$  Decision  $\rightarrow$  State  $\times$  Gain

```

```

DPMModelFunc =
  NotInDomain  $\times$  DecisionSpace  $\times$  Consq
inv (dpm)  $\triangle$  ...

```

where *NotInDomain* restricts the domain of the model (or the problem space), *DecisionSpace* returns the decision space of a given state and *Consq* returns the consequences (ie. gain and succeeding state) of executing a decision.

Based on this data model the basic solution function may be changed into:

```

solution_func:  $DPMModelFunc \times State \rightarrow Gain$ 
solution_func (dpm, st)  $\triangle$ 
  let (not_in_dom, decs, consq) = dpm
  and f (d) =
    let (g, n_st) = consq (st, d)
    in if not_in_dom (n_st) then g
      else comb (g, solution_func (dpm, n_st))
  in minimum (f, decs (st))

```

The flexibility of this model and solution function will be exemplified in section 4.

Selected parts of a SML-implementation (see Paulson (1991) for an introduction) is shown in the appendix. The function *not_in_dom* (of type: *NotInDomain*) is in this implementation a global function and not a part of the data model.

3. GENERALIZED RDP

The basic deterministic model, *DPMModel* is easily generalized to include stochastic processes.

A general summation function used in the definitions is defined as:

```

sum:  $(X \rightarrow Y) \times X\text{-set} \rightarrow Y$ 
sum (f, as)  $\triangle$ 
  let l_sum (s, l_as) =
    if l_as = {} then s
    else let a  $\in$  l_as
          in l_sum (s + f (a), l_as - {a})
  and fst_a  $\in$  as
  in l_sum (f (fst_a), as - {fst_a})
  pre as  $\neq$  {}

```

3.1 The model

Let random events (or states) be represented by the data type *ChanceState*. The general model might then be defined as:

$$Prob = \{ x \in \mathbb{R} \mid 0 < x \leq 1 \}$$

$$GenState = State \cup ChanceState$$

$$DecConsq = Decision \rightarrow Gain \times GenState$$

$$ChanceConsq = (Prob \times Gain \times GenState)\text{-set}$$

$$inv(c) \triangleq \sum(fst, c) = 1$$

$$GenConsq = GenState \rightarrow DecConsq \cup ChanceConsq$$

$$inv(gc) \triangleq \dots$$

$$GDPMModel =$$

$$NotInDomain \times DecisionSpace \times GenConsq$$

$$inv(dpm) \triangleq \dots$$

The graph in fig. 2 is an example of an instance of this model and it illustrates an important feature of the model: Decision nodes and chance nodes might appear in any order. A stochastic DP model (see eg. Nemhauser, 1966) is seen to be a special case of *GDPMModel*, in which a *State* always (unless it is a goal state) is succeeded by *ChanceStates* and vice versa. In this special case the representation of chance states is superfluous and the model might be reformulated as:

$$StochConsq =$$

$$State \rightarrow Decision \rightarrow (Prob \times Gain \times State)\text{-set}$$

$$SDPMModel =$$

$$NotInDomain \times DecisionSpace \times StochConsq$$

$$inv(sdpm) \triangleq \dots$$

Fig. 2. An instance of a General DP model. Nodes represent values (and not variables as in Bayesian Networks).

3.2 The solution function

A solution function for the general model is defined as:

```

solution_gen: GDPMModel × GenState → Gain
solution_gen(gdpm, st) △
  let (not_mem, decs, consq) = gdpm
  in if st ∈ State
     then
       let f(d) =
         let (g, n_st) = consq(d)
         in if not_mem(n_st) then g
            else comb(g,
                      solution_gen(gdpm, n_st))
       in minimum(f, decs(st))
     else
       let f((prob, g, n_st)) =
         if not_mem(n_st) then prob*g
         else prob*
           comb(g, solution_gen(gdpm, n_st))
       in sum(f, consq)

```

which is seen to be an extended version of *solution_func*: if the state *st* is a *ChanceState* the weighted gains are summed up; if not, the gains are (exactly as in *solution_func*) minimized.

4. HEURISTIC RULES AND BOUNDING

Dynamic programming is in itself a very powerful technique, but in some situations heuristic reasoning might (or even has to) be applied in order to reduce state spaces (and to make incomputable problems computable). The solution functions presented in section 2 and 3 might be extended to include heuristic reasoning.

4.1 Heuristic rules

Without significantly effecting the optimal solution a *heuristic rule* for each state in the state space reduces the decision space and consequently the state space itself. These rules (and restrictions on decision spaces in general) are simply included in the models as modifications of *DecisionSpace*.

A road map (continued). The example might illustrate how restrictions on decision spaces might reduce the state space. The rule/restriction applied to the example is as follows: "Always follow the road at the left if you have the option." Fig. 3 illustrates this situation; roads "at the right" have been erased from the original graph (fig. 1) and only one path from A to B remains. The state space has been reduced to 4 states (A, N1, N3 and N5) and the recursive solution function is "reduced" to a simple summation function; *min_two* is never called.

Fig. 3. The graph of fig. 1 partitioned into stages and applied the rule: "always follow the road at the left if you have the option."

Slaughter pig marketing. Another example might illustrate the potential magnitude of the state space reductions: A farmer has a barn which is partitioned into 3 sections; each section containing from 0 to 5 pigs (of same age). Sections has to be emptied after no more than 15 weeks (this is actually a decision space restriction) and sections are assumed equal. Pigs might be marketed at any time. The problem is: when is it time to market individual pigs and when is it time to empty a section and insert a new group of 5 piglets? The size of the state space of a section is $15 \times 6 = 90$ and the size of the state space of the barn pr. stage is: $90 \times 89 \times 88 + 90 \times 1 \times 89 + 90 \times 1 \times 1 = 712,980$. A set of restrictions /heuristic rules might describe the fact that many of these states (like eg. "no pigs in any section") under normal conditions are very rare:

- Max 1 insertion pr. week (labor restrictions).
- No deliveries before week 13 (from experience/growth models).
- One pig is delivered pr. section pr. week from the age of 13 weeks (from experience).

Given these restrictions the total number of states pr. stage is 1,690, but after optimization (40 weeks) the actual maximum number of states per stage is 462.

Value Iteration. In VI or *backward multistage problem solving* (Nemhauser, 1966) the problem space is partitioned into *stages*. Each stage contains problems that might be solved independently and in any order. The stages are solved in an order which guarantees that all sub-problems (ie. problems associated with succeeding states) already have been solved and recorded (problems associated with goal states is initially known). Fig.3 illustrates one valid partitioning of the graph of fig. 1; problems in stage 1 are solved first, followed by problems in stage 2, etc.

Fig. 3 also illustrates one of the main disadvantages of VI: Redundant problem solving. Problems (like those associated with node N2, N4, N6 and N7) which are not on a path leading from the start node will be solved, but they will never be reused. In contrast to RDP (where only problems on paths leading from the start node are

represented and solved) there is in general no easy way (except processing the reduced state space prior to the optimization) to avoid this redundant problem solving in the VI algorithm.

The memory usage due to the recursive nature of the algorithm is the main disadvantage of RDP compared to VI. However, the memory complexity is linear in the depth of the problem or the number of stages and will therefore rarely cause any significant problems.

4.2 Bounding

Heuristic functions (which returns an under (over) estimate of the solution to a minimization (maximization) problem, see eg. Rich and Knight, 1991) might without effecting the solution to the problem reduce the problem space. The problem space is reduced by *bounding* sub-graphs which with certainty will not lead to a better possible global solution than the current best known possible solution.

Kure (1995) defines a solution function for the basic model (*RDPModel*) which utilizes heuristic functions. The function is a special case of an algorithm known in AI as A*: The search strategy is a depth first search instead of an "intelligent" mixture of breadth first and depth first search as in A*.

A similar solution function might be defined for the general DP model and based on that function an A* algorithm for stochastic problems might be derived.

5. CONCLUSION

Several aspects of DP have been examined and discussed in this paper. Some of these need special considerations.

The basic solution function, *solution* described in section 2 is seen to simply be a reformulation (with *comb* substituted by the infix operator '+' and the stage number interpreted as a state variable) of (1). Despite this very close and obvious relationship, the recursive approach towards DP presented in this paper, might seem new to most people dealing with DP and it appears difficult to find the approach described in literature (Kure, 1995).

The stepwise development of solution functions presented in this paper has shown that the use of *memory functions* is a distinct and excludable part of DP. In certain situations it might even appear more (time- and/or space-) efficient to exclude a memory function from the solution function than to include it.

The recursive approach implies easy assessment of heuristic rules (and of decision space restricting rules in general). This feature might appear very beneficial in practical applications: Any appropriate rule is easily applied to the model and only states which according to

the rules are on a path from the start state will be represented and solved. This easy application of heuristic rules is from an applicational point of view the most interesting feature. From a theoretical point of view the flexibility and generality of the models and solution functions is of greater interest and as indicated the approach might lead to a stochastic A* algorithm and/or algorithms for solving problems that traditionally are represented as *Bayesian Networks*.

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APPENDIX: A SML-IMPLEMENTATION

Fragments of program:

```
...
type State = Stage*StateVar
type DPModel = State -> (Decision list
    *(Decision -> (Gain*State)));
...
fun minimum (f, []) = raise Error
  | minimum (f, fst_d::d_set) =
    let
      fun l_min (m, []) = m
        | l_min (m, d::l_d_set) =
            l_min (min_of_two (m,
                f(d)), l_d_set)
    in l_min (f(fst_d), d_set)
    end;

fun solution (dpm, st) =
  let fun f(d) =
        let val (g, n_st) =
            snd(dpm(st)) (d)
        in if n_st not_in_dom dpm
            then g
            else comb (g,
                solution (dpm, n_st))
        end
    in minimum (f, fst(dpm(st)))
    end;
```

Corresponding fragments of output:

```
...
type State = Stage * StateVar
type DPModel = State -> Decision list *
    (Decision -> (Gain * State))
...
val minimum = fn : ('a -> Gain) * ('a -> real) -> real
val solution = fn
  : (int * 'a -> 'b list * ('b -> real
    * (int * 'a))) * (int * 'a) -> real
```